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Isomonodromic deformations with an irregular singularity and the elliptic Θ -function

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Abstract

We will study a monodromy preserving deformation with an irregular singular point and determine the τ -function of the monodromy preserving deformation by the elliptic θ -function moving the argument z and the period Ω .

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1. Introduction

In this paper, we construct a monodromy preserving deformation whose τ -function is represented by the θ -function moving both the argument z and the period Ω . The monodromy preserving deformation has an irregular singularity of Poincaré rank 1.

In [5], Miwa–Jimbo–Ueno extended the work of Schlesinger in [9] and established a general theory of the monodromy preserving deformation for a first-order matrix system of ordinary linear differential equations,

$$\frac{dY}{dx} = A(x)Y, \quad A(x) = \sum_{v=1}^n \sum_{k=0}^{r_v} \frac{A_{v,-k}}{(x - a_v)^{k+1}} - \sum_{k=1}^{r_\infty} A_{\infty,-k} x^{k-1}, \quad (1.1)$$

having regular or irregular singularities of arbitrary rank.

The monodromy data to be preserved are

- (i) Stokes multipliers $S_j^{(v)}$ ($j = 1, \dots, 2r_v$),
- (ii) connection matrices $C^{(v)}$,
- (iii) ‘exponents of formal monodromy’ $T_0^{(v)}$.

Miwa–Jimbo found a deformation equation as a necessary and sufficient condition for the monodromy data to be invariant for deformation parameters and defined the τ -function for the deformation equation.

We explain the relationship between the τ -function and the θ -function. In [6], Miwa–Jimbo expressed the τ -function with the θ -function by moving the argument z of the θ -function.

The monodromy preserving deformation has irregular singularities. The Riemann surface is a ramified covering of \mathbb{CP}^1 with the covering degree m and its genus is g . Especially, the case of genus one is written in [4], which we will describe in the appendix.

In [7], Kitaev–Korotkin calculated the τ -function with the θ -function by moving the period Ω of the θ -function. The monodromy preserving deformation of the τ -function is Fuchsian and has $2g + 2$ regular singular points. The Riemann surface is a hyperelliptic curve with genus g . Especially, the τ -function of genus one is equivalent to the Picard’s solution of the sixth Painlevé equation by the Bäcklund transformation. The aim of this paper is to unify Miwa–Jimbo’s result and Kitaev–Korotkin’s result in the elliptic case. We note that in [1] Deift, Its, Kapaev and Zhou also expressed the τ -function in terms of hyperelliptic functions.

We explain this paper in detail. In section 2, we show some definitions and facts of elliptic functions, which is necessary to construct the τ -function.

In section 3, we will study the following ordinary differential equation:

$$\frac{dY}{dx} = \left(\frac{B_{-1}}{(x - a)^2} + \frac{B_0}{x - a} + \sum_{\nu=1}^3 \frac{A_\nu}{x - e_\nu} \right) Y(x), \tag{1.2}$$

whose deformation parameters are e_1, e_2, e_3 and the diagonal elements of B_{-1} , and define the τ -function.

In section 4, we construct a fundamental solution of (1.2) and calculate the following special monodromy data:

- (i) Stokes multipliers around $x = a$ $S_1^a = S_2^a = 1$,
- (ii) connection matrices around $x = e_\nu$ $C^{(\nu)} (\nu = 1, 2, 3, \infty)$,
- (iii) the exponents of formal monodromy $T_0^a = 0, T_0^{(\nu)} = \text{diag}(-\frac{1}{4}, \frac{1}{4}) (\nu = 1, 2, 3, \infty)$,

where e_∞ is ∞ .

In section 5, we concretely calculate the monodromy preserving deformation from the fundamental solution and compute the coefficients $B_{-1}, B_0, A_\nu (\nu = 1, 2, 3)$.

In section 6, we calculate the τ -function. Section 6 consists of three subsections. Subsection 6.1 is devoted to H_t , the Hamiltonian on the deformation parameter t . Subsections 6.2 and 6.3 are about $H_\nu (\nu = 1, 2, 3)$, the Hamiltonians on the deformation parameters $e_\nu (\nu = 1, 2, 3)$. In subsection 6.2, we show some facts about elliptic functions in order to calculate the Hamiltonians $H_\nu (\nu = 1, 2, 3)$. In subsection 6.3, we calculate $H_\nu (\nu = 1, 2, 3)$ and the τ -function. Our main theorem is as follows:

Theorem 1.1. *For the monodromy preserving deformation (1.2), the τ -function is*

$$\begin{aligned} \tau(e_1, e_2, e_3, t) &= \theta[p, q] \left(\frac{t}{\omega_1}; \Omega \right) \omega_1^{-\frac{1}{2}} \prod_{1 \leq \nu < \mu \leq 3} (e_\nu - e_\mu)^{-\frac{1}{8}} \\ &\quad \times \exp \left(\frac{\eta_1 t^2}{2\omega_1} \right) \exp \left(\frac{t^2}{4} f(e_1, e_2, e_3) \right), \end{aligned}$$

where

$$f(e_1, e_2, e_3) = -a + \frac{1}{3} \sum_{\nu=1}^3 e_\nu + \frac{1}{2} \prod_{\nu=1}^3 (a - e_\nu) \sum_{\mu=1}^3 \frac{1}{(a - e_\mu)^2}.$$

Remark. $\theta[p, q](z; \Omega)$ is defined in (2.8). By setting $t = 0$, we get a τ -function which is essentially the same as Kitaev–Korotkin’s τ -function in [7]. We have a branch point at $x = \infty$, while Kitaev–Korotkin does not.

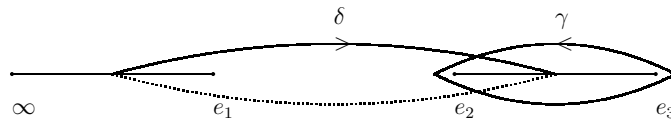


Figure 1. Branch cuts and canonical basis of cycles on the elliptic curve, E . Continuous and dashed paths lie on the first and second sheets of E , respectively.

2. Elliptic function theory

We review the elliptic function theory to fix our notation (see [8]). We consider the elliptic curve E defined by the equation

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3)$$

with arbitrary constants $e_i \in \mathbb{C}$ ($e_i \neq e_j$), where we do not assume that $e_1 + e_2 + e_3 = 0$. We set the new coordinate

$$\tilde{x} = x - \frac{1}{3} \sum_{i=1}^3 e_i, \tilde{e}_j = e_j - \frac{1}{3} \sum_{i=1}^3 e_i \quad \text{for } j = 1, 2, 3$$

and define the Abel map in the following way:

$$u := \int_{\infty}^x \frac{dx}{\sqrt{4(x - e_1)(x - e_2)(x - e_3)}} = \int_{\infty}^{\tilde{x}} \frac{d\tilde{x}}{\sqrt{4(\tilde{x} - \tilde{e}_1)(\tilde{x} - \tilde{e}_2)(\tilde{x} - \tilde{e}_3)}}. \tag{2.1}$$

Let us define periods ω_1, ω_2 by

$$\omega_1 = \int_{\gamma} \frac{dx}{y}, \quad \omega_2 = \int_{\delta} \frac{dx}{y}. \tag{2.2}$$

From the bilinear relation of Riemann, we get

$$\Omega := \frac{\omega_2}{\omega_1}, \quad \text{Im}(\Omega) > 0. \tag{2.3}$$

$\tilde{x} = \tilde{x}(u)$, the inverse function of the Abel map, can be expressed with the Weierstrass \wp -function:

$$x - \frac{1}{3} \sum_{i=1}^3 e_i = \tilde{x}(u) = \wp(u; \omega_1, \omega_2). \tag{2.4}$$

$\wp(u)$ satisfies the following differential equations:

$$\wp'(u)^2 = 4(\wp(u) - \tilde{e}_1)(\wp(u) - \tilde{e}_2)(\wp(u) - \tilde{e}_3) \tag{2.5}$$

$$= 4\wp(u)^3 - g_2\wp(u) - g_3. \tag{2.6}$$

We have

$$\sigma(u; \omega_1, \omega_2) = \exp\left(\frac{\eta_1}{2\omega_1} u^2\right) \frac{\omega_1}{\theta'_{11}} \theta_{11}\left(\frac{u}{2\omega_1}; \Omega\right). \tag{2.7}$$

We set the σ -function with characteristic $p, q \in \mathbb{C}$ in the following way:

$$\sigma[p, q](u) := \exp\left(\frac{\eta_1}{2\omega_1} u^2\right) \frac{\omega_1}{\theta'_{11}} \theta[p, q]\left(\frac{u}{\omega_1}; \Omega\right), \tag{2.8}$$

where

$$\theta[p, q](z; \Omega) := \sum_{n \in \mathbb{Z}} \exp 2\pi i \left\{ \frac{1}{2} \Omega (n+p)^2 + (n+p)(z+q) \right\}. \quad (2.9)$$

$\sigma[p, q](z)$ has the following periodicity properties:

$$\sigma[p, q](u + \omega_1) = \exp(2\pi i p) \exp \eta_1 \left(u + \frac{\omega_1}{2} \right) \sigma[p, q](u), \quad (2.10)$$

$$\sigma[p, q](u + \omega_2) = \exp(-2\pi i q) \exp \eta_2 \left(u + \frac{\omega_2}{2} \right) \sigma[p, q](u). \quad (2.11)$$

3. The degenerated Schlesinger system

We study the following ordinary differential equation:

$$\frac{dY}{dx} = \left(\frac{B_{-1}}{(x-a)^2} + \frac{B_0}{x-a} + \sum_{v=1}^3 \frac{A_v}{x-e_v} \right) Y(x), \quad (3.1)$$

where $A_1, A_2, A_3, B_{-1}, B_0 \in sl(2, \mathbb{C})$ are independent of x .

The monodromy data of 3.1 are as follows:

- (i) Stokes multipliers around $x = a$ $S_1^a = S_2^a$,
- (ii) connection matrices around ∞, e_v $C^\infty, C^{(v)}$ ($v = 1, 2, 3$),
- (iii) ‘the exponents of formal monodromy’ $T_0^a, T_0^{(\infty)}, T_0^{(v)}$ ($v = 1, 2, 3$).

In the next section, as the solutions of (3.1), we will get convergent series around $x = a, \infty, e_v$ ($v = 1, 2, 3$) in the following way:

$$Y(x) = (1 + O(x-a)) \exp T^{(a)}(x) = \hat{Y}^{(a)}(x) \exp T^{(a)}(x), \quad (3.2)$$

$$Y(x) = G^{(\infty)} \left(1 + O\left(\frac{1}{x}\right) \right) \exp T^{(\infty)}(x) = G^{(\infty)} \hat{Y}^\infty(x) \exp T^{(\infty)}(x), \quad (3.3)$$

$$Y(x) = G^{(v)} (1 + O(x-e_v)) \exp T^{(v)}(x) = G^{(v)} \hat{Y}^{(v)}(x) \exp T_0^{(v)}(x), \quad (3.4)$$

$$(v = 1, 2, 3). \quad (3.5)$$

where

$$T^{(a)}(x) = \begin{pmatrix} \frac{\wp'(\alpha)t}{2} & \\ & -\frac{\wp'(\alpha)t}{2} \end{pmatrix} \frac{(x-a)^{-1}}{-1} + T_0^{(a)} \log(x-a), \quad (3.6)$$

$$T^{(\infty)}(x) = T_0^\infty \log\left(\frac{1}{x}\right), \quad (3.7)$$

$$T^{(v)}(x) = T_0^{(v)} \log(x-e_v) \quad (v = 1, 2, 3). \quad (3.8)$$

The deformation parameters of (3.1) are e_1, e_2, e_3 , and the diagonal elements of B_{-1}, t . For e_1, e_2, e_3, t we set the following closed 1-form:

$$\omega = \omega_a + \omega_{e_1} + \omega_{e_2} + \omega_{e_3} \quad (3.9)$$

$$= H_t dt + H_1 de_1 + H_2 de_2 + H_3 de_3, \quad (3.10)$$

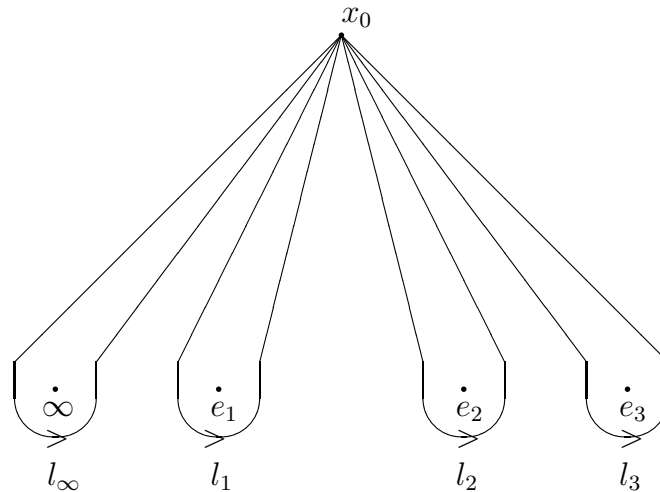


Figure 2. Generators of $\pi_1(\mathbb{CP}^1 \setminus \{\infty, e_1, e_2, e_3\}, x_0)$.

where

$$\omega_a = -\text{Res}_{x=a} \text{tr} \hat{Y}^{(a)}(x)^{-1} \frac{\partial \hat{Y}^{(a)}}{\partial x}(x) dT^{(a)}(x), \tag{3.11}$$

$$\omega_{e_\nu} = -\text{Res}_{x=e_\nu} \text{tr} \hat{Y}^{(\nu)}(x)^{-1} \frac{\partial \hat{Y}^{(\nu)}}{\partial x}(x) dT^{(\nu)}(x) \quad (\nu = 1, 2, 3), \tag{3.12}$$

and d is the exterior differentiation with respect to the deformation parameters e_1, e_2, e_3, t . Especially, we can write

$$\omega_{e_\nu} = \left[\text{Res}_{x=e_\nu} \frac{1}{2} \text{tr} \left(\frac{dY}{dx} Y^{-1} \right)^2 \right] de_\nu. \tag{3.13}$$

From the closed 1-form ω , we define the τ -function in the following way:

$$\omega := d \log \tau(e_1, e_2, e_3, t). \tag{3.14}$$

4. The Riemann–Hilbert problem for special parameters

In this section, we concretely construct a (2×2) -matrix-valued function $Y(x)$, whose monodromy data, (i) Stokes multipliers, (ii) connection matrices, (iii) ‘exponents of formal monodromies’, are independent of the deformation parameters t, e_1, e_2, e_3 .

We denote the Abel map $u = u(x)$ by $u = u(P)$, because we regard it as a function of the Riemann surface E defined by

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3). \tag{4.1}$$

We set the involution of E in the following way:

$$* : (x, y) \longrightarrow (x, -y). \tag{4.2}$$

We define the (2×2) -matrix-valued function $\Phi(P)$ with the σ -function and the ζ -function in the following way:

$$\Phi(P) = \begin{pmatrix} \varphi(P) & \varphi(P^*) \\ \psi(P) & \psi(P^*) \end{pmatrix}, \tag{4.3}$$

where

$$\begin{aligned}\varphi(P) &= \sigma[p, q](u + u_\varphi + t)\sigma(u - u_\varphi) \exp -\frac{t}{2}\{\zeta(u - \alpha) + \zeta(u + \alpha)\} \\ \psi(P) &= \sigma[p, q](u + u_\psi + t)\sigma(u - u_\psi) \exp -\frac{t}{2}\{\zeta(u - \alpha) + \zeta(u + \alpha)\} \\ u_\varphi &= u(P_\varphi), \quad u_\psi = u(P_\psi), \quad \alpha = u(a),\end{aligned}$$

with arbitrary $P_\varphi, P_\psi \in E$.

Proposition 4.1. *The function $\Phi(P)$ is holomorphic and invertible outside of the branch points ∞, e_1, e_2, e_3 and transforms as follows with respect to the tracing along the basic cycles γ, δ :*

$$\begin{aligned}T_\gamma[\Phi(P)] &= \Phi(P) \exp\{\eta_1(2u + \omega_1)\} \exp \pi i(2p + 1)\sigma_3 \\ T_\delta[\Phi(P)] &= \Phi(P) \exp\{\eta_2(2u + \omega_2)\} \exp -\pi i(2q + 1)\sigma_3\end{aligned}$$

where by T_l we denote the operator of analytic continuation along the contour l and set Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proof. By using the periodicity (2.10) and (2.11), we get

$$\begin{aligned}T_\gamma[\Phi(P)] &= \Phi(P) \exp\{\eta_1(2u + \omega_1)\} \exp \pi i(2p + 1)\sigma_3 \\ T_\delta[\Phi(P)] &= \Phi(P) \exp\{\eta_2(2u + \omega_2)\} \exp -\pi i(2q + 1)\sigma_3.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}T_\gamma[\det \Phi(P)] &= \exp 2\eta_1(2u + \omega_1) \det \Phi(P) \\ T_\delta[\det \Phi(P)] &= \exp 2\eta_2(2u + \omega_2) \det \Phi(P).\end{aligned}$$

By integrating $\frac{d(\det \Phi(P))}{\det \Phi(P)}$ along the fundamental polygon of E and using the Legendre relation, we get

$$\frac{1}{2\pi i} \oint_{\partial \hat{E}} \frac{d(\det \Phi(P))}{\det \Phi(P)} = \frac{1}{2\pi i} 4(\eta_1\omega_2 - \eta_2\omega_1) = 4. \quad (4.4)$$

Therefore, four zeros of $\det \Phi(P)$ on E are branch points ∞, e_1, e_2, e_3 . We proved the proposition. \square

We normalize $\Phi(P)$ at $x = a$. We can easily show the local behavior of the Abel map.

Lemma 4.2

$$u - \alpha = \frac{1}{\wp'(\alpha)}(x - a) - \frac{\wp''(\alpha)}{2\wp'''(\alpha)}(x - a)^2 + \left(\frac{\wp''(\alpha)^2}{2\wp'(\alpha)^5} - \frac{\wp'''(\alpha)}{6\wp'(\alpha)^4} \right) (x - a)^3 + \dots \quad (4.5)$$

From lemma 4.2, $\Phi(x)$ can be developed in the following way:

$$\Phi(x) = (G^{(a)} + O(x - a)) \exp -\frac{\wp'(\alpha)t}{2(x - a)}\sigma_3,$$

where $G^{(a)}$ is a 2×2 matrix with the matrix elements

$$\begin{aligned} (G^{(a)})_{11} &= \sigma[p, q](\alpha + u_\varphi + t)\sigma(\alpha - u_\varphi) \exp \left\{ -\frac{t}{2} \left(\frac{\wp''(\alpha)}{2\wp'(\alpha)} + \zeta(2\alpha) \right) \right\}, \\ (G_0^{(a)})_{21} &= \sigma[p, q](\alpha + u_\psi + t)\sigma(\alpha - u_\psi) \exp \left\{ -\frac{t}{2} \left(\frac{\wp''(\alpha)}{2\wp'(\alpha)} + \zeta(2\alpha) \right) \right\}, \\ (G^{(a)})_{12} &= \sigma[p, q](-\alpha + u_\varphi + t)\sigma(-\alpha - u_\varphi) \exp \left\{ \frac{t}{2} \left(\frac{\wp''(\alpha)}{2\wp'(\alpha)} + \zeta(2\alpha) \right) \right\}, \\ (G^{(a)})_{22} &= \sigma[p, q](-\alpha - u_\psi + t)\sigma(-\alpha - u_\psi) \exp \left\{ \frac{t}{2} \left(\frac{\wp''(\alpha)}{2\wp'(\alpha)} + \zeta(2\alpha) \right) \right\}. \end{aligned}$$

From proposition 4.1,

$$\det G^{(a)} = \det \Phi(a) \neq 0.$$

We define a matrix-valued function $Y(x)$ in the following way:

$$Y(P) = \frac{\sqrt{\det \Phi(a)}}{\sqrt{\det \Phi(P)}} (G^{(a)})^{-1} \Phi(P), \tag{4.6}$$

By the normalization near $x = a$, we obtain the following lemma:

Lemma 4.3

$$\begin{aligned} Y(x) &= (1 + O(x - a)) \exp T^{(a)}(x) \\ T^{(a)}(x) &= T_{-1}^{(a)} \frac{(x - a)^{-1}}{-1} \\ T_{-1}^{(a)} &= \frac{\wp'(\alpha)t}{2} \sigma_3. \end{aligned}$$

Especially, if $u_\varphi = \alpha, u_\psi = -\alpha$,

$$\begin{aligned} Y(x) &= (1 + Y_1^{(a)}(x - a) + \dots) \exp T^{(a)}(x) \\ (Y_1^{(a)})_{11} &= \frac{1}{\wp'(\alpha)} \left\{ \frac{\sigma[p, q]'(t)}{\sigma[p, q](t)} - \frac{t}{2} \left(4\wp(\alpha) - \frac{1}{2} \left(\frac{\wp''(\alpha)}{\wp'(\alpha)} \right)^2 \right) \right\} \\ (Y_1^{(a)})_{21} &= \frac{\sigma[p, q](2\alpha + t)}{\sigma[p, q](t)\sigma[p, q](2\alpha)\wp'(\alpha)} \exp \left\{ -t \left(\frac{\wp''(\alpha)}{2\wp'(\alpha)} + \zeta(2\alpha) \right) \right\} \\ (Y_1^{(a)})_{12} &= \frac{-\sigma[p, q](-2\alpha + t)}{\sigma[p, q](t)\sigma[p, q](2\alpha)\wp'(\alpha)} \exp \left\{ t \left(\frac{\wp''(\alpha)}{2\wp'(\alpha)} + \zeta(2\alpha) \right) \right\} \\ (Y_1^{(a)})_{22} &= \frac{-1}{\wp'(\alpha)} \left\{ \frac{\sigma[p, q]'(t)}{\sigma[p, q](t)} - \frac{t}{2} \left(4\wp(\alpha) - \frac{1}{2} \left(\frac{\wp''(\alpha)}{\wp'(\alpha)} \right)^2 \right) \right\}. \end{aligned}$$

$Y(x)$ has the following monodromy and Stokes matrices.

Theorem 4.4. For $v = \infty, 1, 2, 3$, we set

$$M_v = \begin{pmatrix} 0 & m_v \\ -m_v^{-1} & 0 \end{pmatrix}. \tag{4.7}$$

The matrix elements corresponding to the contour l_v ($v = \infty, 1, 2, 3$) are given by the expressions

$$\begin{aligned} m_\infty &= -i, & m_1 &= i \exp(-2\pi i p) \\ m_2 &= -i \exp 2\pi i(-p + q) \\ m_3 &= i \exp(2\pi i q). \end{aligned}$$

Stokes matrices are as follows:

$$S_1 = S_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. From proposition 4.1, we note that the branch points ∞, e_1, e_2, e_3 on E are zero points of $\det \Phi(P)$ of the first order. According to proposition 4.1, we obtain

$$\begin{aligned} T_\gamma[Y(x)] &= Y(x)M_3M_2 \\ &= Y(x) \left(\exp \left\{ \frac{2\pi i}{4} \right\} \right)^{-2} \exp \pi i(2p+1)\sigma_3 \\ &= Y(x) \exp 2\pi i p \sigma_3, \\ T_{\delta^{-1}}[Y(x)] &= Y(x)M_2M_1 \\ &= Y(x) \left(\exp \left\{ \frac{2\pi i}{4} \right\} \right)^{-2} \exp \pi i(2q+1)\sigma_3 \\ &= Y(x) \exp 2\pi i q \sigma_3. \end{aligned}$$

By the involution $*$, we get

$$Y(x)M_\infty = iY(x)\sigma_1 \quad \text{or} \quad = -iY(x)\sigma_1.$$

We set

$$m_\infty = -i.$$

From the property of the monodromy group, we obtain

$$M_3M_2M_1 = M_\infty^{-1}. \quad (4.8)$$

From

$$\begin{aligned} M_3M_2 &= \text{diag}(\exp\{2\pi i p\}, \exp\{-2\pi i p\}), \\ M_2M_1 &= \text{diag}(\exp\{2\pi i q\}, \exp\{-2\pi i q\}), \end{aligned}$$

we get

$$\begin{aligned} M_1 &= \begin{pmatrix} & -m_\infty \exp\{-2\pi i p\} \\ m_\infty^{-1} \exp\{2\pi i p\} & \end{pmatrix}, \\ M_3 &= \begin{pmatrix} & -m_\infty \exp\{2\pi i q\} \\ m_\infty^{-1} \exp\{-2\pi i q\} & \end{pmatrix}. \end{aligned}$$

Then we get

$$M_2 = \begin{pmatrix} & m_\infty \exp\{2\pi i(q-p)\} \\ -m_\infty^{-1} \exp\{2\pi i(p-q)\} & \end{pmatrix}$$

from (4.8).

From lemma 4.3, the Stokes matrices are

$$S_1^{(a)} = S_2^{(a)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

□

We can state the monodromy data of $Y(x)$ in the following way.

Corollary 4.5. $Y(x)$ has the following monodromy data:

(i) Stokes multipliers $S_1^a = S_2^a = 1$,

- (ii) connection matrices $C^{(v)} = \frac{1}{\sqrt{2im_v}} \begin{pmatrix} i & -m_v \\ & m_v \end{pmatrix} (v = \infty, 1, 2, 3)$,
- (iii) formal monodromies $T_0^{(v)} = \text{diag}(-\frac{1}{4}, \frac{1}{4}) (v = \infty, 1, 2, 3)$.

Epecially, the behaviors of $Y(x)$ near $e_v (v = \infty, 1, 2, 3)$ are

$$\begin{aligned} Y(x) &= G^{(v)}(1 + O(x_v)) \exp T^{(v)}(x)C^{(v)} \\ T^{(v)}(x) &= T_0^{(v)}(x) \log x_v \\ x_\infty &= x^{-1}, \quad x_v = x - e_v. \end{aligned}$$

Proof. (i) is clear. (ii) and (iii) can be obtained by diagonalizing the monodromy matrices $M_v (v = \infty, 1, 2, 3)$. □

This corollary means that the monodromy data of $Y(x)$ are independent of the deformation parameters t, e_1, e_2, e_3 .

5. Monodromy preserving deformation

In this section, we prove that $Y(x)$ satisfies an ordinary differential equation of the following type:

$$\frac{dY}{dx} = \left(\frac{B_{-1}}{(x-a)^2} + \frac{B_0}{x-a} + \sum_{v=1}^3 \frac{A_v}{x-e_v} \right) Y(x),$$

and concretely determine the coefficients $B_{-1}, B_0, A_v (v = 1, 2, 3)$.

Firstly, we define symbols in order to describe the monodromy preserving deformation. For $v = 1, 2, 3$, we set

$$\begin{aligned} D^{(v)}(u) &= \frac{2m_v}{m_\infty} \varphi(u)\psi(u) \left(\frac{d}{du} \log \varphi(u) - \frac{d}{du} \log \psi(u) \right), \\ \tilde{\omega}_1 &= \frac{\omega_1}{2}, \quad \tilde{\omega}_2 = \frac{\omega_1 + \omega_2}{2}, \quad \tilde{\omega}_3 = \frac{\omega_2}{2}, \\ \tilde{\eta}_1 &= \eta_1, \quad \tilde{\eta}_2 = \eta_1 + \eta_2, \quad \tilde{\eta}_3 = \eta_2. \end{aligned}$$

We note that the monodromy matrices and the Stoke matrices do not have the parameters P_φ, P_ψ in theorem 4.4. Then, in this section, we set $u_\varphi = \alpha, u_\psi = -\alpha$ from the uniqueness of the Riemann–Hilbert problem.

Theorem 5.1. *$Y(x)$ satisfies the following ordinary differential equation:*

$$\frac{dY}{dx} = \left(\frac{B_{-1}}{(x-a)^2} + \frac{B_0}{x-a} + \sum_{v=1}^3 \frac{A_v}{x-e_v} \right) Y(x), \tag{5.1}$$

where

$$B_{-1} = \text{diag} \left(\frac{\wp'(\alpha)t}{2}, -\frac{\wp'(\alpha)t}{2} \right)$$

$$B_0 = \text{diag}(0, 0)$$

$$A_v = G^{(v)} \text{diag} \left(-\frac{1}{4}, \frac{1}{4} \right) (G^{(v)})^{-1} \quad (v = 1, 2, 3),$$

$$G^{(v)} = \begin{pmatrix} 0 & \exp \left\{ \frac{t}{2} \left(\frac{\wp''(\alpha)}{2\wp'(\alpha)} + \zeta(2\alpha) \right) \right\} \\ -\exp \left\{ -\frac{t}{2} \left(\frac{\wp''(\alpha)}{2\wp'(\alpha)} + \zeta(2\alpha) \right) \right\} & 0 \end{pmatrix}$$

$$\begin{aligned} &\times \frac{\sqrt{2m_\nu}}{\sqrt{D^{(\nu)}(\tilde{\omega}_\nu)} i} \begin{pmatrix} \varphi(\tilde{\omega}_\nu) & \varphi(\tilde{\omega}_\nu) \left(\frac{d}{du} \log \varphi(\tilde{\omega}_\nu) - \tilde{\eta}_\nu\right) \\ \psi(\tilde{\omega}_\nu) & \psi(\tilde{\omega}_\nu) \left(\frac{d}{du} \log \psi(\tilde{\omega}_\nu) - \tilde{\eta}_\nu\right) \end{pmatrix} \\ &\times \begin{pmatrix} \left(\frac{\varphi'(\tilde{\omega}_\nu)}{2}\right)^{\frac{1}{4}} & \\ & \left(\frac{\varphi'(\tilde{\omega}_\nu)}{2}\right)^{-\frac{1}{4}} \end{pmatrix}. \end{aligned}$$

The behavior near ∞ is

$$Y(x) = G^{(\infty)}(1 + O(x^{-1})) \exp T^{(\infty)}(x),$$

where

$$\begin{aligned} G^{(\infty)} &= \begin{pmatrix} 0 & \exp\left\{\frac{t}{2}\left(\frac{\varphi''(\alpha)}{2\varphi'(\alpha)} + \zeta(2\alpha)\right)\right\} \\ \exp\left\{-\frac{t}{2}\left(\frac{\varphi''(\alpha)}{2\varphi'(\alpha)} + \zeta(2\alpha)\right)\right\} & 0 \end{pmatrix} \\ &\times \frac{1}{\left(\frac{d}{du} \log \varphi(0) - \frac{d}{du} \log \psi(0)\right)^{\frac{1}{2}}} \begin{pmatrix} -i\varphi(0) & i\varphi'(0) \\ -i\psi(0) & i\psi'(0) \end{pmatrix}. \end{aligned}$$

Proof. From lemma 4.3, we get

$$\begin{aligned} Y'(x)Y^{-1}(x) &= \frac{1}{(x-a)^2} \begin{pmatrix} \frac{\varphi'(\alpha)t}{2} & \\ & -\frac{\varphi'(\alpha)t}{2} \end{pmatrix} + \text{regular part} \\ &:= \frac{B_{-1}}{(x-a)^2} + \frac{B_0}{x-a} + \text{regular part}. \end{aligned}$$

Therefore, B_{-1}, B_0 are determined as in theorem 5.1.

In the following, we determine $G^{(\nu)}$. We represent the monodromy matrices M_ν ($\nu = 1, 2, 3$) in the following way:

$$M_\nu = C^{(\nu)-1} - i\sigma_3 C^{(\nu)} = (C^{(\nu)})^{-1} \begin{pmatrix} \exp\left\{-\frac{2\pi i}{4}\right\} & \\ & \exp\left\{\frac{2\pi i}{4}\right\} \end{pmatrix} C^{(\nu)},$$

where $C^{(\nu)}$ is the connection matrix. We set $G^{(\nu)}$ ($\nu = 1, 2, 3$) in the following way:

$$Y(x) = G^{(\nu)}(1 + O(x - e_\nu))(x - e_\nu)^{-\frac{1}{4}\sigma_3} C^{(\nu)}. \tag{5.2}$$

In order to calculate $G^{(\nu)}$, we compute the constant term of

$$\begin{aligned} Y(x)(C^{(\nu)})^{-1}(x - e_\nu)^{\frac{1}{4}\sigma_3} &= Y(x) \frac{1}{\sqrt{2im_\nu}} \begin{pmatrix} m_\nu & m_\nu \\ -i & i \end{pmatrix} (x - e_\nu)^{\frac{1}{4}\sigma_3} \\ &= \frac{\sqrt{\det \Phi(a)}}{\sqrt{\det \Phi(u)}} (G^{(a)})^{-1} \\ &\times \frac{1}{\sqrt{2im_\nu}} \begin{pmatrix} m_\nu \varphi(u) - i\varphi(-u) & m_\nu \varphi(u) + i\varphi(-u) \\ m_\nu \psi(u) - i\psi(-u) & m_\nu \psi(u) + i\psi(-u) \end{pmatrix} (x - e_\nu)^{\frac{1}{4}\sigma_3}. \end{aligned}$$

By setting $u_\varphi = \alpha, u_\psi = -\alpha$ in $G^{(a)}$,

$$\sqrt{\det \Phi(a)}(G^{(a)})^{-1} = \begin{pmatrix} 0 & \exp\left\{\frac{t}{2}\left(\frac{\varphi''(\alpha)}{2\varphi'(\alpha)} + \zeta(2\alpha)\right)\right\} \\ -\exp\left\{-\frac{t}{2}\left(\frac{\varphi''(\alpha)}{2\varphi'(\alpha)} + \zeta(2\alpha)\right)\right\} & 0 \end{pmatrix}. \tag{5.3}$$

By direct computation, we get

$$\begin{aligned} \det \Phi(u) &= (u - \tilde{\omega}_\nu)(\varphi'(\tilde{\omega}_\nu)\psi(-\tilde{\omega}_\nu) - \varphi(\tilde{\omega}_\nu)\psi'(-\tilde{\omega}_\nu)) \\ &\quad + \varphi'(-\tilde{\omega}_\nu)\psi(\tilde{\omega}_\nu) - \varphi(-\tilde{\omega}_\nu)\psi'(\tilde{\omega}_\nu) + \dots \\ &= (u - \tilde{\omega}_\nu) \left\{ \varphi(\tilde{\omega}_\nu)\psi(-\tilde{\omega}_\nu) \left(\frac{\varphi'(\tilde{\omega}_\nu)}{\varphi(\tilde{\omega}_\nu)} - \frac{\psi'(-\tilde{\omega}_\nu)}{\psi(-\tilde{\omega}_\nu)}\right) \right. \\ &\quad \left. + \varphi(-\tilde{\omega}_\nu)\psi(\tilde{\omega}_\nu) \left(\frac{\varphi'(-\tilde{\omega}_\nu)}{\varphi(-\tilde{\omega}_\nu)} - \frac{\psi'(\tilde{\omega}_\nu)}{\psi(\tilde{\omega}_\nu)}\right) \right\} + \dots \end{aligned}$$

From (2.10), (2.11),

$$\begin{aligned} \det \Phi(u) &= (u - \tilde{\omega}_v) \frac{m_v}{m_\infty} \varphi(\tilde{\omega}_v) \psi(\tilde{\omega}_v) \times 2 \left(\frac{d}{du} \log \varphi(\tilde{\omega}_v) - \frac{d}{du} \log \psi(\tilde{\omega}_v) \right) + O((u - \tilde{\omega}_v)^2) \\ &= (u - \tilde{\omega}_v) D^{(v)}(\tilde{\omega}_v) + O((u - \tilde{\omega}_v)^2) \\ &= (x - e_v)^{\frac{1}{2}} \left(\frac{\wp''(\tilde{\omega}_v)}{2} \right)^{-\frac{1}{2}} D^{(v)}(\tilde{\omega}_v) + O((\sqrt{x - e_v})^2). \end{aligned}$$

From (2.10), (2.11), we note that

$$m_v \varphi(\tilde{\omega}_v) + i\varphi(-\tilde{\omega}_v) = 0, \quad m_v \psi(\tilde{\omega}_v) + i\psi(-\tilde{\omega}_v) = 0.$$

Then,

$$\begin{aligned} G^{(v)} &= Y(x)(C^{(v)})^{-1}(x - e_v)^{\frac{1}{4}\sigma_3} \Big|_{x=e_v} \\ &= (\det \Phi(a))^{\frac{1}{2}} (G^{(a)})^{-1} (D^{(v)}(\tilde{\omega}_v))^{-\frac{1}{2}} \left(\frac{\wp''(\tilde{\omega}_v)}{2} \right)^{\frac{1}{4}} \\ &\quad \times \frac{1}{\sqrt{2im_v}} \begin{pmatrix} m_v \varphi(u) - i\varphi(-u) & \frac{m_v \varphi(u) + i\varphi(-u)}{u - \tilde{\omega}_v} \\ m_v \psi(u) - i\psi(-u) & \frac{m_v \psi(u) + i\psi(-u)}{u - \tilde{\omega}_v} \end{pmatrix} \begin{pmatrix} 1 & \\ & \frac{u - \tilde{\omega}_v}{(x - e_v)^{\frac{1}{2}}} \end{pmatrix} \Big|_{x=e_v} \\ &= (\det(a))^{\frac{1}{2}} (G^{(a)})^{-1} \frac{\sqrt{2m_v}}{\sqrt{D^{(v)}(\tilde{\omega}_v)} i} \begin{pmatrix} \varphi(\tilde{\omega}_v) & \varphi(\tilde{\omega}_v) \left(\frac{d}{du} \log \varphi(\tilde{\omega}_v) - \tilde{\eta}_v \right) \\ \psi(\tilde{\omega}_v) & \psi(\tilde{\omega}_v) \left(\frac{d}{du} \log \psi(\tilde{\omega}_v) - \tilde{\eta}_v \right) \end{pmatrix} \\ &\quad \times \begin{pmatrix} \left(\frac{\wp''(\tilde{\omega}_v)}{2} \right)^{\frac{1}{4}} & \\ & \left(\frac{\wp''(\tilde{\omega}_v)}{2} \right)^{-\frac{1}{4}} \end{pmatrix}. \end{aligned}$$

From (5.2), we get

$$\begin{aligned} Y'(x)Y^{-1}(x) &= \frac{1}{(x - e_v)} G^{(v)} \left(-\frac{1}{4} \right) \sigma_3 (G^{(v)})^{-1} + \text{regular part} \\ &:= \frac{A_v}{(x - e_v)} + \text{regular part}. \end{aligned}$$

We set $G^{(\infty)}$ in the following way:

$$Y(x) = G^{(\infty)}(1 + O(x^{-1})) \exp T^{(\infty)}(x).$$

In the same way, we can compute $G^{(\infty)}$. □

Remark. For the simplification, we do not normalize B_{-1} .

We can get the following deformation equation from theorem 5.1.

Corollary 5.2. *The deformation equation of the monodromy preserving deformation (5.1) is as follows. For $v = 1, 2, 3$,*

$$\begin{aligned} dA_v &= \sum_{\mu \neq v} \frac{[A_\mu, A_v]}{e_v - e_\mu} de_\mu + \sum_{\mu} \frac{[A_\mu, A_v]}{a - e_\mu} de_\mu - \frac{[A_v, B_{-1}]}{(a - e_v)^2} de_v \\ &\quad + \frac{[dT_{-1}^{(a)}, A_v]}{a - e_v} + [[dT_{-1}^{(a)}, Y_1^{(a)}], A_v], \end{aligned} \tag{5.4}$$

where d is the exterior differentiation with respect to the deformation parameters, t, e_1, e_2, e_3 and $T_{-1}^{(a)}, Y_1^{(a)}$ is defined in lemma 4.3.

6. The τ -function for the degenerated Schlesinger system

In this section, we will calculate the τ -function for (5.1). This section consists of three subsections. Subsection 6.1 is devoted to the Hamiltonian H_t . Subsections 6.2 and 6.3 are devoted to the Hamiltonian H_ν ($\nu = 1, 2, 3$). In subsection 6.2, we show some facts about elliptic functions. In subsection 6.3, we calculate H_ν and the τ -function.

6.1. The Hamiltonian at the irregular singular point

In this subsection, we compute ω_a , H_t in the following way:

Proposition 6.1

$$\omega_a = \frac{\sigma[p, q]'(t)}{\sigma[p, q](t)} dt + \frac{\sigma[p, q]'(t)}{\sigma[p, q](t)} t \left(-\frac{de_1}{2(a-e_1)} - \frac{de_2}{2(a-e_2)} - \frac{de_3}{2(a-e_3)} \right) + f(e_1, e_2, e_3) \left(\frac{t dt}{2} - \frac{t^2 de_1}{4(a-e_1)} - \frac{t^2 de_2}{4(a-e_2)} - \frac{t^2 de_3}{4(a-e_3)} \right),$$

where

$$f(e_1, e_2, e_3) = -a + \frac{1}{3} \sum_{\nu=1}^3 e_\nu + \frac{1}{2} \prod_{\nu=1}^3 (a - e_\nu) \sum_{\mu=1}^3 \frac{1}{(a - e_\mu)^2}. \quad (6.1)$$

From ω_a , we get

$$\begin{aligned} H_t &= \frac{\sigma[p, q]'(t)}{\sigma[p, q](t)} + \frac{t}{2} f(e_1, e_2, e_3) \\ &= \frac{\partial}{\partial t} \left\{ \frac{\eta_1 t^2}{2\omega_1} \right\} + \frac{\partial}{\partial t} \left\{ \log \theta[p, q] \left(\frac{t}{\omega_1} \right) \right\} + \frac{\partial}{\partial t} \left\{ \frac{t^2}{4} f(e_1, e_2, e_3) \right\}. \end{aligned}$$

Proof. From the definition of ω_a ,

$$\omega_a = -\text{Res}_{x=a} \text{tr} \hat{Y}^{(a)}(x)^{-1} \frac{\partial \hat{Y}^{(a)}}{\partial x}(x) dT^{(a)}(x). \quad (6.2)$$

We set

$$\Pi(P) := -\frac{t}{2} \{ \zeta(u - \alpha) + \zeta(u + \alpha) \} \quad (6.3)$$

From lemma 4.2, we get

$$\begin{aligned} \Pi(P) &= -\frac{t}{2} \left\{ \frac{1}{u - \alpha} + \dots + \zeta(u + \alpha) \right\} \\ &= -\frac{\wp'(\alpha)t}{2} \frac{1}{(x - a)} - \frac{t}{2} \left(\frac{\wp''(\alpha)}{2\wp'(\alpha)} + \zeta(2\alpha) \right) \\ &\quad - \frac{t}{2} \left(-\frac{\wp(2\alpha)}{\wp'(\alpha)} - \frac{\wp''(\alpha)^2}{4\wp'(\alpha)^3} + \frac{\wp'''(\alpha)}{6\wp'(\alpha)^2} \right) (x - a) + \dots \end{aligned}$$

We denote the regular part of $\Pi(P)$ around $x = a$ by $\hat{\Pi}(P)$:

$$\hat{\Pi}(P) := \Pi(P) - \left(-\frac{\wp'(\alpha)t}{2} \frac{1}{(x - a)} \right). \quad (6.4)$$

We set

$$\hat{\varphi}(P) = \sigma[p, q](u + u_\varphi + t)\sigma(u - u_\varphi), \quad \hat{\psi}(P) = \sigma[p, q](u + u_\psi + t)\sigma(u - u_\psi).$$

Then, we get

$$\begin{aligned} Y(P) &= \frac{\sqrt{\det \Phi(a)}}{\sqrt{\det \Phi(P)}} (G^{(a)})^{-1} \Phi(P) \\ &= \frac{\sqrt{\det \Phi(a)}}{\sqrt{\det \Phi(P)}} (G^{(a)})^{-1} \begin{pmatrix} \hat{\varphi}(P) \exp(\hat{\Pi}(P)) & \hat{\varphi}(P^*) \exp(\hat{\Pi}(P^*)) \\ \hat{\psi}(P) \exp(\hat{\Pi}(P)) & \hat{\psi}(P^*) \exp(\hat{\Pi}(P^*)) \end{pmatrix} \\ &\quad \times \text{diag} \left(\exp \left\{ -\frac{\wp'(\alpha)t}{2} \frac{1}{x-a} \right\}, \exp \left\{ \frac{\wp'(\alpha)t}{2} \frac{1}{x-a} \right\} \right) \\ &:= \hat{Y}^{(a)}(x) \exp T^{(a)}(x) \end{aligned}$$

According to the definition of ω_a ,

$$\omega_a = -\text{Res}_{x=a} \text{tr} \hat{Y}^{(a)}(x)^{-1} \frac{\partial}{\partial x} \hat{Y}^{(a)}(x) dT^{(a)}(x). \tag{6.5}$$

We set

$$A(x) = (G^{(a)})^{-1} \begin{pmatrix} \hat{\varphi}(u) \exp \hat{\Pi}(u) & \hat{\varphi}(-u) \exp \hat{\Pi}(-u) \\ \hat{\psi}(u) \exp \hat{\Pi}(u) & \hat{\psi}(-u) \exp \hat{\Pi}(-u) \end{pmatrix}. \tag{6.6}$$

By direct calculation, we get

$$\begin{aligned} -\text{tr} \hat{Y}^{(a)}(x)^{-1} \frac{\partial}{\partial x} \hat{Y}^{(a)}(x) dT^{(a)}(x) \\ = \frac{d(\wp'(\alpha)t)}{2} \frac{1}{(x-a)} \text{tr} A^{-1}(x) A'(x) \sigma_3. \end{aligned}$$

We obtain

$$\begin{aligned} \text{tr} A^{-1}(x) A'(x) \sigma_3 &= \frac{1}{\det \Phi(u)} \left[\det \begin{pmatrix} \{\hat{\varphi}(u) \exp \hat{\pi}(u)\}' & \hat{\varphi}(-u) \exp \hat{\pi}(-u) \\ \{\hat{\psi}(u) \exp \hat{\pi}(u)\}' & \hat{\psi}(-u) \exp \hat{\pi}(-u) \end{pmatrix} \right. \\ &\quad \left. - \det \begin{pmatrix} \hat{\varphi}(u) \exp \hat{\pi}(u) & \{\hat{\varphi}(-u) \exp \hat{\pi}(-u)\}' \\ \hat{\psi}(u) \exp \hat{\pi}(u) & \{\hat{\psi}(-u) \exp \hat{\pi}(-u)\}' \end{pmatrix} \right], \end{aligned}$$

where \prime means the differentiation with respect to the variable x . We normalized the matrix function $Y(x)$ around $x = a$ like lemma 4.3 and proved that the monodromy group of $Y(x)$ is independent of P_φ, P_ψ in theorem 4.4 and its corollary. Therefore, we can choose the parameters P_φ, P_ψ at our disposal to simplify the calculation. Firstly, we multiply both the numerators and the denominators by $\frac{1}{x_\psi - x_\varphi}$. We take the limit $P_\psi \rightarrow P_\varphi$ and get

$$\hat{\psi}(P) = \frac{\partial \hat{\varphi}(P)}{\partial x_\varphi}. \tag{6.7}$$

Secondly, we multiply both the numerators and the denominators by $\frac{1}{x_\varphi - x}$. We take the limit $P_\varphi \rightarrow P$ and obtain

$$\text{tr} A^{-1}(x) A'(x) \sigma_3 = 2 \frac{1}{\sigma[p, q](t)} \lim_{P_\varphi \rightarrow P} \frac{\partial}{\partial x_\varphi} \sigma[p, q](-u + u_\varphi + t) + 2 \frac{\partial}{\partial x} \{\hat{\Pi}(P)\}. \tag{6.8}$$

According to the definition of ω_a ,

$$\begin{aligned} \omega_a &= \frac{d(\wp'(\alpha)t)}{2} \text{tr} A^{-1}(x) A'(x) \sigma_3|_{x=a} \\ &= \frac{d(\wp'(\alpha)t)}{2} \left(2 \frac{\sigma[p, q]'(t)}{\sigma[p, q](t)} \frac{1}{y} + \frac{\partial}{\partial x} \hat{\Pi}(P) \right) \Big|_{x=a} \\ &= \frac{d(\wp'(\alpha)t)}{\wp'(\alpha)} \frac{\sigma[p, q]'(t)}{\sigma[p, q](t)} + d(\wp'(\alpha)t) \left\{ -\frac{t}{2} \left(-\frac{\wp(2\alpha)}{\wp'(\alpha)} - \frac{\wp''(\alpha)^2}{4\wp'(\alpha)^3} + \frac{\wp'''(\alpha)}{6\wp'(\alpha)^2} \right) \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{\sigma[p, q]'(t)}{\sigma[p, q](t)} dt + \frac{\sigma[p, q]'(t)}{\sigma[p, q](t)} t \frac{d\wp'(\alpha)}{\wp'(\alpha)} \\
&\quad + \frac{t}{2} dt \left(\wp(2\alpha) + \frac{\wp''(\alpha)^2}{4\wp'(\alpha)^2} - \frac{\wp'''(\alpha)}{6\wp'(\alpha)} \right) + \frac{t^2}{2} \frac{d\wp'(\alpha)}{\wp'(\alpha)} \left(\wp(2\alpha) + \frac{\wp''(\alpha)^2}{4\wp'(\alpha)^2} - \frac{\wp'''(\alpha)}{6\wp'(\alpha)} \right),
\end{aligned}$$

where d is the exterior differentiation with respect to the deformation parameters e_1, e_2, e_3, t . In order to compute ω_a , we need some preparations about the \wp -function. By direct calculation, we get

$$\frac{d\wp'(\alpha)}{\wp'(\alpha)} = -\frac{1}{2} \frac{1}{a-e_1} de_1 - \frac{1}{2} \frac{1}{a-e_2} de_2 - \frac{1}{2} \frac{1}{a-e_3} de_3. \quad (6.9)$$

We use the addition theorem

$$\wp(2\alpha) = -2\wp(\alpha) + \frac{1}{4} \left(\frac{\wp''(\alpha)}{\wp'(\alpha)} \right)^2, \quad (6.10)$$

and the following equation:

$$\wp'''(\alpha) = 12\wp'(\alpha)\wp(\alpha). \quad (6.11)$$

From the relationship between the \wp -function and the Abel-map, we obtain

$$\begin{aligned}
\wp(\alpha) &= a - \frac{1}{3} \sum_{i=1}^3 e_i \\
\wp'(\alpha)^2 &= 4(a-e_1)(a-e_2)(a-e_3) \\
\wp''(\alpha) &= \sum_{i<j} 2(a-e_i)(a-e_j).
\end{aligned}$$

By using the above equations, we get

$$\begin{aligned}
\omega_a &= \frac{\sigma[p, q]'(t)}{\sigma[p, q](t)} dt + \frac{\sigma[p, q]'(t)}{\sigma[p, q](t)} t \left(-\frac{de_1}{2(a-e_1)} - \frac{de_2}{2(a-e_2)} - \frac{de_3}{2(a-e_3)} \right) \\
&\quad + f(e_1, e_2, e_3) \left(\frac{t dt}{2} - \frac{t^2 de_1}{4(a-e_1)} - \frac{t^2 de_2}{4(a-e_2)} - \frac{t^2 de_3}{4(a-e_3)} \right). \quad \square
\end{aligned}$$

6.2. Three lemmas about elliptic functions

We show three lemmas about elliptic functions.

Lemma 6.2.

$$\omega_1 \eta_1 = -\frac{1}{3} \frac{\theta_{11}'''}{\theta_{11}'^2} \quad (6.12)$$

$$-\frac{1}{3} \left(\sum_{i=1}^3 e_i \right)^2 + (e_1 e_2 + e_2 e_3 + e_3 e_1) = \frac{1}{2} \left(\frac{\theta_{11}^{(5)}}{\theta_{11}^{(1)}} \right) - \frac{5}{6} \left(\frac{\theta_{11}^{(3)}}{\theta_{11}^{(1)}} \right)^2 \frac{1}{\omega_1^4}. \quad (6.13)$$

Proof. See p 410 in [3]. The second equation (6.13) can be proved by comparing the coefficients of u^5 of (2.7) and by using the following equation:

$$\tilde{e}_1 \tilde{e}_2 + \tilde{e}_2 \tilde{e}_3 + \tilde{e}_3 \tilde{e}_1 = -\frac{1}{3} \left(\sum_{i=1}^3 e_i \right)^2 + (e_1 e_2 + e_2 e_3 + e_3 e_1). \quad (6.14)$$

□

Lemma 6.3. For $\nu = 1, 2, 3$, the dependence of the period Ω on the branch points is described by the equation

$$\frac{\partial \Omega}{\partial e_\nu} = \frac{\pi i}{\omega_1^2 \prod_{\mu \neq \nu} (e_\nu - e_\mu)}. \tag{6.15}$$

Proof. See [7] and pp 45, 46 in [2]. □

Lemma 6.4. For $\nu = 1, 2, 3$,

$$\frac{\partial}{\partial e_\nu} \left(\frac{\eta_1 t^2}{2\omega_1} \right) = t^2 \left(\frac{\partial}{\partial e_\nu} \log \omega_1 \right)^2 \prod_{\mu \neq \nu} (e_\nu - e_\mu) - \frac{t^2}{12}. \tag{6.16}$$

Proof. By differentiating the logarithm of both sides of the formula

$$\sqrt[4]{16(e_1 - e_2)^2(e_1 - e_3)^2(e_2 - e_3)^2} = \frac{2\pi}{\omega_1^3} (\theta'_{11})^2,$$

we get

$$3 \frac{\frac{\partial \omega_1}{\partial e_1}}{\omega_1} + \frac{1}{2} \frac{1}{(e_1 - e_2)} + \frac{1}{2} \frac{1}{(e_1 - e_3)} = 2 \frac{\frac{\partial \theta'_{11}}{\partial e_1}}{\theta'_{11}}. \tag{6.17}$$

From the heat equation

$$\frac{\partial \theta[p, q]}{\partial z^2}(z; \Omega) = 4\pi i \frac{\partial \theta[p, q]}{\partial \Omega}(z; \Omega), \tag{6.18}$$

lemmas 6.2, 6.3 and (6.17), we get the lemma. □

6.3. The τ -function

In order to compute ω_{e_ν} ($\nu = 1, 2, 3$), we show the following lemma.

Lemma 6.5. For $\nu = 1, 2, 3$,

$$\begin{aligned} \text{Res}_{x=e_\nu} \frac{1}{2} \left(\frac{dY}{dx} Y^{-1}(x) \right)^2 &= -\frac{1}{8} \left(\sum_{\mu \neq \nu} \frac{1}{e_\nu - e_\mu} \right) - \frac{1}{2} \frac{\partial}{\partial e_\nu} \log \omega_1 \\ &+ t^2 \left(\frac{\partial}{\partial e_\nu} \log \omega_1 \right)^2 \prod_{\mu \neq \nu} (e_\nu - e_\mu) + \frac{\partial}{\partial e_\nu} \left\{ \log \theta[p, q] \left(\frac{t}{\omega_1}; \Omega \right) \right\} \\ &+ \frac{t}{2(a - e_\nu)} \frac{\sigma[p, q]'(\frac{t}{\omega_1}; \Omega)}{\sigma[p, q](\frac{t}{\omega_1}; \Omega)} + \frac{t^2 \prod_{\mu \neq \nu} (e_\nu - e_\mu)}{4(a - e_\nu)^2} \\ &+ \frac{t^2}{6(a - e_\nu)} \left(\sum_{\mu \neq \nu} (e_\nu - e_\mu) \right). \end{aligned}$$

Proof. We prove the case $\nu = 1$. The other cases can be shown in the same way. By direct calculation, we obtain

$$\frac{1}{2} \text{tr} (Y'(x)Y^{-1}(x))^2 = -\frac{\det(\Phi_x)}{\det \Phi} + \frac{1}{4} \left(\frac{(\det \Phi)_x}{\det \Phi} \right)^2.$$

We calculate ω_1 just like ω_a in proposition 6.1. We multiply both the numerators and the denominators of

$$\frac{\det(\Phi_x)}{\det \Phi}, \quad \frac{(\det \Phi)_x}{\det \Phi}$$

by $\frac{1}{x_\varphi - x_\psi}$. We take the limit $P_\psi \rightarrow P_\varphi$ and get

$$\psi(P) = \frac{\partial \varphi(P)}{\partial x_\varphi}. \quad (6.19)$$

We take the limit $P_\varphi \rightarrow P$ and obtain

$$\begin{aligned} \frac{(\det \Phi)_x}{\det \Phi} &= 2 \frac{1}{\sigma(-2u)} \lim_{P_\varphi \rightarrow P} \frac{\partial}{\partial x} \sigma(-u - u_\varphi) \\ \frac{\det(\Phi_x)}{\det \Phi} &= \frac{1}{\sigma[p, q](t)} \lim_{P_\varphi \rightarrow P} \frac{\partial^2}{\partial x \partial x_\varphi} \sigma[p, q](-u + u_\varphi + t) \\ &\quad + \frac{2}{\sigma[p, q](t)} \lim_{P_\varphi \rightarrow P} \frac{\partial}{\partial x} \sigma[p, q](-u + u_\varphi + t) \frac{\partial}{\partial x} \Pi(P) \\ &\quad + \frac{1}{\sigma(-2u)} \lim_{P_\varphi \rightarrow P} \frac{\partial^2}{\partial x \partial x_\varphi} \sigma(-u - u_\varphi) - \left(\frac{\partial}{\partial x} \Pi(P) \right)^2. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \frac{1}{2} \operatorname{tr}(Y'(x)Y^{-1}(x))^2 &= - \lim_{P_\varphi \rightarrow P} \frac{\partial^2}{\partial x \partial x_\varphi} \log \sigma(-u - u_\varphi) \\ &\quad - \frac{1}{\sigma[p, q](t)} \lim_{P_\varphi \rightarrow P} \frac{\partial^2}{\partial x \partial x_\varphi} \sigma[p, q](-u + u_\varphi + t) \\ &\quad - \frac{2}{\sigma[p, q](t)} \lim_{P_\varphi \rightarrow P} \frac{\partial}{\partial x} \sigma[p, q](-u + u_\varphi + t) \frac{\partial}{\partial x} \Pi(P) + \left(\frac{\partial}{\partial x} \Pi(P) \right)^2. \end{aligned}$$

In order to calculate the first term of $\frac{1}{2} \operatorname{tr}(Y'(x)Y^{-1}(x))^2$, we use the following formulae:

$$\begin{aligned} \wp(u) &= - \frac{d^2}{du^2} \log \sigma(u) \\ \wp(u) &= x - \frac{1}{3} \sum_{\mu=1}^3 e_\mu, \quad \tilde{e}_v = e_v - \frac{1}{3} \sum_{\mu=1}^3 e_\mu \quad (v = 1, 2, 3) \\ \wp'(u)^2 &= 4(\wp(u) - \tilde{e}_1)(\wp(u) - \tilde{e}_2)(\wp(u) - \tilde{e}_3) \end{aligned}$$

Then, we get

$$\lim_{P_\varphi \rightarrow P} \frac{\partial^2}{\partial x \partial x_\varphi} \log \sigma(-u - u_\varphi) = \frac{1}{24} \sum_{v=1}^3 \frac{1}{(x - e_v)} \sum_{\mu \neq v} \frac{1}{(e_v - e_\mu)} - \frac{1}{16} \sum_{v=1}^3 \frac{1}{(x - e_v)^2}.$$

By direct calculation, we can calculate the second term of $\frac{1}{2} \operatorname{tr}(Y'(x)Y^{-1}(x))^2$ in the following way:

$$\begin{aligned} \frac{1}{\sigma[p, q](t)} \lim_{P_\varphi \rightarrow P} \frac{\partial^2}{\partial x_\varphi \partial x} \sigma[p, q](-u + u_\varphi + t) \\ = - \frac{\sigma[p, q]''(t)}{\sigma[p, q](t)} \left(\sum_{v=1}^3 \frac{1}{x - e_v} \frac{1}{4 \prod_{\mu \neq v} (e_v - e_\mu)} \right). \end{aligned} \quad (6.20)$$

In order to calculate the third and fourth terms of $\frac{1}{2}\text{tr}(Y'(x)Y^{-1}(x))^2$, we use the following relation:

$$\wp(u) = -\frac{d}{du}\zeta(u). \tag{6.21}$$

By using (6.21), we get

$$\begin{aligned} & \frac{2}{\sigma[p, q](t)} \lim_{P_\varphi \rightarrow P} \frac{\partial}{\partial x} \sigma[p, q](-u + u_\varphi + t) \frac{\partial}{\partial x} \Pi(P) \\ &= t \frac{\sigma[p, q]'(t)}{\sigma[p, q](t)} \left\{ \frac{x}{2 \prod_{v=1}^3 (x - e_v)} + \frac{a}{2 \prod_{v=1}^3 (x - e_v)} - \frac{1}{3} \frac{\sum_{v=1}^3 e_v}{\prod_{v=1}^3 (x - e_v)} \right. \\ & \quad \left. - \frac{1}{2(x-a)^2} - \frac{1}{2} \frac{1}{(x-a)^2} \frac{\prod_{v=1}^3 (a - e_v)}{\prod_{v=1}^3 (x - e_v)} \right\} \end{aligned} \tag{6.22}$$

and

$$\begin{aligned} \text{Res}_{x=e_1} \left(\frac{\partial}{\partial x} \Pi(P) \right)^2 &= \frac{t^2 e_1 - e_2}{36 e_1 - e_3} + \frac{t^2 e_1 - e_3}{36 e_1 - e_2} + \frac{t^2 (e_1 - e_2)(e_1 - e_3)}{4 (a - e_1)^2} \\ & \quad + \frac{t^2}{18} + \frac{t^2 e_1 - e_3}{6 a - e_1} + \frac{t^2 e_1 - e_2}{6 a - e_1}. \end{aligned} \tag{6.23}$$

From (6.20), (6.20), (6.22), (6.23), we get

$$\begin{aligned} \text{Res}_{x=e_1} \frac{1}{2} \text{tr}(Y'(x)Y^{-1}(x))^2 &= -\frac{1}{24} \sum_{j \neq 1} \frac{1}{e_1 - e_j} + \frac{\sigma[p, q]''(t)}{\sigma[p, q](t)} \frac{1}{4(e_1 - e_2)(e_1 - e_3)} \\ & \quad + t \frac{\sigma[p, q]'(t)}{\sigma[p, q](t)} \left(\frac{1}{6} \sum_{j \neq 1} \frac{1}{e_1 - e_j} \right) + t \frac{\sigma[p, q]'(t)}{\sigma[p, q](t)} \frac{1}{2} \frac{1}{(a - e_1)} + \frac{t^2 e_1 - e_3}{36 e_1 - e_2} \\ & \quad + \frac{t^2 e_1 - e_2}{36 e_1 - e_3} + \frac{t^2 (e_1 - e_2)(e_1 - e_3)}{4 (a - e_1)^2} + \frac{t^2}{18} + \frac{t^2 e_1 - e_3}{6 a - e_1} + \frac{t^2 e_1 - e_2}{6 a - e_1}. \end{aligned}$$

By directly calculating the third term of $\text{Res}_{x=e_1} \frac{1}{2} \text{tr}(Y'(x)Y^{-1}(x))^2$, we get

$$\begin{aligned} t \frac{\sigma[p, q]'(t)}{\sigma[p, q](t)} \left(\frac{1}{6} \sum_{j \neq 1} \frac{1}{e_1 - e_j} \right) &= \frac{\eta_1 t^2}{6\omega_1} \left(\frac{1}{e_1 - e_2} + \frac{1}{e_1 - e_3} \right) \\ & \quad + \frac{t}{6\omega_1} \frac{\theta[p, q]'(\frac{t}{\omega_1}; \Omega)}{\theta[p, q](\frac{t}{\omega_1}; \Omega)} \left(\frac{1}{e_1 - e_2} + \frac{1}{e_1 - e_3} \right). \end{aligned}$$

From lemmas 6.2, (6.18) and (6.17), we obtain

$$\begin{aligned} \frac{\eta_1 t^2}{6\omega_1} \left(\frac{1}{e_1 - e_2} + \frac{1}{e_1 - e_3} \right) &= -\frac{t^2}{3} \frac{\partial \omega_1}{\partial e_1} (e_1 - e_2) - \frac{t^2}{3} \frac{\partial \omega_1}{\partial e_1} (e_1 - e_3) \\ & \quad - \frac{t^2}{9} - \frac{t^2 e_1 - e_2}{18 e_1 - e_3} - \frac{t^2 e_1 - e_3}{18 e_1 - e_2}. \end{aligned}$$

By directly calculating the second term of $\text{Res}_{x=e_1} \frac{1}{2} \text{tr}(Y'(x)Y^{-1}(x))^2$, we have

$$\begin{aligned} \frac{\sigma[p, q]''(t)}{\sigma[p, q](t)} \frac{1}{4(e_1 - e_2)(e_1 - e_3)} &= \left(\frac{\eta_1}{\omega_1}\right) \frac{1}{4(e_1 - e_2)(e_1 - e_3)} \\ &+ \left(\frac{\eta_1 t}{\omega_1}\right)^2 \frac{1}{4(e_1 - e_2)(e_1 - e_3)} + \frac{2\eta_1 t}{\omega_1^2} \frac{\theta[p, q]'(\frac{t}{\omega_1}; \Omega)}{\theta[p, q](\frac{t}{\omega_1}; \Omega)} \frac{1}{4(e_1 - e_2)(e_1 - e_3)} \\ &+ \frac{1}{\omega_1^2} \frac{\theta[p, q]''(\frac{t}{\omega_1}; \Omega)}{\theta[p, q](\frac{t}{\omega_1}; \Omega)} \frac{1}{4(e_1 - e_2)(e_1 - e_3)}. \end{aligned}$$

From lemmas 6.2, (6.18) and (6.17), each of the four terms of

$$\frac{\sigma[p, q]''(t)}{\sigma[p, q](t)} \frac{1}{4(e_1 - e_2)(e_1 - e_3)}$$

is as follows:

$$\left(\frac{\eta_1}{\omega_1}\right) \frac{1}{4(e_1 - e_2)(e_1 - e_3)} = -\frac{1}{2} \frac{\frac{\partial \omega_1}{\partial e_1}}{\omega_1} - \frac{1}{12} \frac{1}{(e_1 - e_2)} - \frac{1}{12} \frac{1}{(e_1 - e_3)}, \quad (6.24)$$

$$\begin{aligned} \left(\frac{\eta_1 t}{\omega_1}\right)^2 \frac{1}{4(e_1 - e_2)(e_1 - e_3)} &= t^2 \left(\frac{\frac{\partial \omega_1}{\partial e_1}}{\omega_1}\right)^2 (e_1 - e_2)(e_1 - e_3) + \frac{t^2}{36} \frac{e_1 - e_3}{e_1 - e_2} + \frac{t^2}{36} \frac{e_1 - e_2}{e_1 - e_3} \\ &+ \frac{t^2}{3} \left(\frac{\frac{\partial \omega_1}{\partial e_1}}{\omega_1}\right) (e_1 - e_2) + \frac{t^2}{3} \left(\frac{\frac{\partial \omega_1}{\partial e_1}}{\omega_1}\right) (e_1 - e_3) + \frac{t^2}{18} \end{aligned} \quad (6.25)$$

$$\begin{aligned} \frac{2\eta_1 t^2}{\omega_1^2} \frac{\theta[p, q]'(\frac{t}{\omega_1}; \Omega)}{\theta[p, q](\frac{t}{\omega_1}; \Omega)} \frac{1}{4(e_1 - e_2)(e_1 - e_3)} &= -\frac{t}{6\omega_1} \frac{1}{(e_1 - e_2)} \frac{\theta[p, q]'(\frac{t}{\omega_1}; \Omega)}{\theta[p, q](\frac{t}{\omega_1}; \Omega)} \\ &- \frac{t}{6\omega_1} \frac{1}{(e_1 - e_3)} \frac{\theta[p, q]'(\frac{t}{\omega_1}; \Omega)}{\theta[p, q](\frac{t}{\omega_1}; \Omega)} - \frac{\frac{\partial \omega_1}{\partial e_1}}{\omega_1^2} t \frac{\theta[p, q]'(\frac{t}{\omega_1}; \Omega)}{\theta[p, q](\frac{t}{\omega_1}; \Omega)}, \end{aligned} \quad (6.26)$$

and

$$\begin{aligned} \frac{1}{\omega_1^2} \frac{\theta[p, q]''(\frac{t}{\omega_1}; \Omega)}{\theta[p, q](\frac{t}{\omega_1}; \Omega)} \frac{1}{4(e_1 - e_2)(e_1 - e_3)} &= \left(\frac{\frac{\partial}{\partial \Omega} \theta[p, q](\frac{t}{\omega_1}; \Omega)}{\theta[p, q](\frac{t}{\omega_1}; \Omega)}\right) \left(\frac{\pi i}{\omega_1^2 (e_1 - e_2)(e_1 - e_3)}\right) \\ &= \frac{\frac{\partial}{\partial \Omega} \theta[p, q](\frac{t}{\omega_1}; \Omega)}{\theta[p, q](\frac{t}{\omega_1}; \Omega)} \frac{\partial \Omega}{\partial e_1}. \end{aligned} \quad (6.27)$$

We note that the sum of (6.26) and (6.27) is

$$\begin{aligned} \frac{\partial}{\partial e_1} \left\{ \log \theta[p, q] \left(\frac{t}{\omega_1}; \Omega\right) \right\} - \frac{t}{6\omega_1} \frac{1}{(e_1 - e_2)} \frac{\theta[p, q]'(\frac{t}{\omega_1}; \Omega)}{\theta[p, q](\frac{t}{\omega_1}; \Omega)} \\ - \frac{t}{6\omega_1} \frac{1}{(e_1 - e_3)} \frac{\theta[p, q]'(\frac{t}{\omega_1}; \Omega)}{\theta[p, q](\frac{t}{\omega_1}; \Omega)}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \text{Res}_{x=e_1} \frac{1}{2} (Y'(x)Y^{-1}(x))^2 &= -\frac{1}{8} \left(\sum_{\mu \neq 1} \frac{1}{e_1 - e_\mu} \right) - \frac{1}{2} \frac{\partial}{\partial e_1} \log \omega_1 \\ &+ t^2 \left(\frac{\partial}{\partial e_1} \log \omega_1 \right)^2 \prod_{\mu \neq 1} (e_1 - e_\mu) + \frac{\partial}{\partial e_1} \left\{ \log \theta[p, q] \left(\frac{t}{\omega_1}; \Omega\right) \right\} \end{aligned}$$

$$\begin{aligned}
 & + \frac{t}{2(a - e_1)} \frac{\sigma[p, q]'(\frac{t}{\omega_1}; \Omega)}{\sigma[p, q](\frac{t}{\omega_1}; \Omega)} + \frac{t^2 \prod_{\mu \neq 1} (e_1 - e_\mu)}{4(a - e_1)^2} \\
 & + \frac{t^2}{6(a - e_1)} \left(\sum_{\mu \neq 1} (e_1 - e_\mu) \right).
 \end{aligned}$$

□

We obtain ω_a in proposition 6.1 and get ω_{e_v} in lemma 6.5. Therefore, we have H_v ($v = 1, 2, 3$) in the following way:

Proposition 6.6. For $v = 1, 2, 3$,

$$\begin{aligned}
 H_v = \frac{\partial}{\partial e_v} \left\{ \log \theta[p, q] \left(\frac{t}{\omega_1}; \Omega \right) \right\} & - \frac{1}{2} \frac{\partial}{\partial e_v} \{ \log \omega_1 \} - \frac{1}{8} \sum_{\mu \neq v} \frac{1}{(e_v - e_\mu)} \\
 & + \frac{\partial}{\partial e_v} \left(\frac{\eta_1 t^2}{2\omega_1} \right) + \frac{\partial}{\partial e_v} \left\{ \frac{t^2}{4} f(e_1, e_2, e_3) \right\}.
 \end{aligned}$$

Since we have calculated the Hamiltonians H_t, H_1, H_2, H_3 , we finally get the following theorem:

Theorem 6.7. For the monodromy preserving deformation (5.1), the τ -function is

$$\begin{aligned}
 \tau(e_1, e_2, e_3; t) = \theta[p, q] \left(\frac{t}{\omega_1}; \Omega \right) & \omega_1^{-\frac{1}{2}} \prod_{1 \leq v < \mu \leq 3} (e_v - e_\mu)^{-\frac{1}{8}} \\
 & \times \exp \left(\frac{\eta_1 t^2}{2\omega_1} \right) \exp \left(\frac{t^2}{4} f(e_1, e_2, e_3) \right),
 \end{aligned}$$

where

$$f(e_1, e_2, e_3) = -a + \frac{1}{3} \sum_{v=1}^3 e_v + \frac{1}{2} \prod_{v=1}^3 (a - e_v) \sum_{\mu=1}^3 \frac{1}{(a - e_\mu)^2}.$$

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Appendix A

In [4], Miwa–Jimbo expressed the τ -function in terms of the Weierstrass σ -function. The paper, however, is written in Japanese. For the reader’s convenience we summarize in English those results in this appendix. Here, we assume that $e_1 + e_2 + e_3 = 0$. We fix a point α in one dimension complex torus \mathbb{T}^1 with period ω_1, ω_2 . We take $l \in \mathbb{Z}$ and the parameter $t \in \mathbb{C}$. We consider the following function:

$$\begin{aligned}
 y_l(z) & = y_l(z, t, \alpha) \\
 & = \sigma(z - \alpha)^{l-1} \sigma(z + \alpha)^{-l} \sigma(z + t + (2l - 1)\alpha) \exp \left\{ -\frac{t}{2} (\zeta(u - \alpha) + \zeta(u + \alpha)) \right\}.
 \end{aligned}$$

From the quasi-periodicity of the σ -function and the ζ -function, $y_l(z)$ is holomorphic in \mathbb{T} except for α . Near $z = \alpha$,

$$\begin{aligned}
 y_l(z) &= \hat{y}_l(z)(z - \alpha)^{l-1} \exp\left(-\frac{t}{2}\left(\frac{1}{z - \alpha} - \frac{\wp''(\alpha)}{2\wp'(\alpha)}\right)\right) \\
 \hat{y}_l(z) &= c_0(1 + c_1(z - \alpha) + \dots) \\
 c_0 &= \sigma(2\alpha)^{-l} \sigma(t + 2l\alpha) \exp\left(-\frac{t}{2}\zeta(2\alpha) - \frac{t}{4}\frac{\wp''(\alpha)}{\wp'(\alpha)}\right) = c_0(t, \alpha, l) \\
 c_1 &= \zeta(t + 2l\alpha) - l\zeta(2\alpha) + \frac{t}{2}\wp(2\alpha) = c_1(t, \alpha, l).
 \end{aligned}$$

The above property characterizes $y_l(z)$.

We set $e_1, e_2, e_3, e_\infty = \infty \in \mathbb{P}^1$ as the branch points of the covering map $\mathbb{T} \rightarrow \mathbb{P}^1, z \rightarrow x = \wp(z)$. We define the multi-valued analytic matrix $Y(x)$ in the following way:

$$\begin{aligned}
 Y(x) &= G_0^{-1} \tilde{Y}(z) \\
 G_0^{-1} &= \text{diag}\left(\frac{1}{c_0(t, \alpha, l)}, \frac{1}{c_0(-t, \alpha, -l)}\right) \\
 \tilde{Y}(z) &= \begin{pmatrix} y_l(z) & y_l(-z) \\ y_{l+1}(z) & y_{l+1}(-z) \end{pmatrix}.
 \end{aligned}$$

We assume $\wp(\alpha) \neq e_1, e_2, e_3, e_\infty$. Then $Y(x)$ satisfies the following linear ordinary differential equation:

$$\begin{aligned}
 \frac{dY}{dx} &= A(x)Y \\
 A(x) &= \frac{A_{-2}}{(x - a)^2} + \frac{A_{-1}}{x - a} + \frac{B_1}{x - e_1} + \frac{B_2}{x - e_2} + \frac{B_3}{x - e_3} \\
 A_{-2} &= \begin{pmatrix} \frac{i}{2} & \\ & -\frac{i}{2} \end{pmatrix}, \quad \bar{i} = \wp'(\alpha)t, \quad a = \wp(\alpha).
 \end{aligned}$$

In order to check this, we will study the monodromy of $Y(x)$.

A.1. The behavior near the irregular singular point

We note that

$$z - \alpha = \frac{1}{\wp'(\alpha)}(x - a) - \frac{\wp''(\alpha)}{2\wp'''(\alpha)}(x - a)^2 + \left(\frac{\wp''(\alpha)^2}{2\wp'(\alpha)^5} - \frac{\wp'''(\alpha)}{6\wp'(\alpha)^4}\right)(x - a)^3 + \dots \quad (\text{A.1})$$

We get the following equations:

$$\begin{aligned}
 Y(x) &= \hat{Y}^{(\alpha)}(x) \exp\{T^{(\alpha)}(x) + K\} \\
 T^{(\alpha)}(x) &= \begin{pmatrix} \frac{i}{2} & \\ & -\frac{i}{2} \end{pmatrix} \frac{-1}{x - a} + \begin{pmatrix} l - 1 & \\ & -l - 1 \end{pmatrix} \log(x - a) \\
 K &= \begin{pmatrix} -l + 1 & \\ & l + 1 \end{pmatrix} \log \wp'(\alpha) \\
 \hat{Y}^{(\alpha)}(x) &= \left(1 + Y_1^{(\alpha)}(x - a) + \dots\right) \\
 Y_1^{(\alpha)} &= \frac{1}{\wp'(\alpha)} \tilde{Y}_1^{(\alpha)} + \frac{t}{2} \left(\frac{\wp''(\alpha)^2}{4\wp'(\alpha)^3} - \frac{\wp'''(\alpha)}{6\wp'(\alpha)^2}\right) \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} - \frac{\wp''(\alpha)}{\wp'(\alpha)^2} \begin{pmatrix} l - 1 & \\ & -l - 1 \end{pmatrix} \\
 \hat{Y}_1^{(\alpha)} &= \begin{pmatrix} c_1(t, \alpha, l) & \frac{c_0(-t, \alpha, 1-l)}{c_0(t, \alpha, l)} \\ \frac{c_0(t, \alpha, l+1)}{c_0(-t, \alpha, -l)} & c_1(-t, \alpha, -l) \end{pmatrix}.
 \end{aligned}$$

A.2. The behavior near regular singular points

We assume $\nu = 1, 2, 3$. By direct calculation, we get

$$y_l(z; t, \alpha) = y_l(\tilde{\omega}_\nu)(1 + f_l^{(\nu)}(z - \tilde{\omega}_\nu) + \dots)$$

$$y_l(\tilde{\omega}_\nu; t, \alpha) = \frac{\sigma_\nu(t + (2l - 1)\alpha)}{\sigma_\nu(\alpha)}$$

$$f_l(\tilde{\omega}_\nu) = \zeta_\nu(t + (2l - 1)\alpha) - \zeta_\nu(\tilde{\omega}_\nu + \alpha) + t\wp(\alpha + \tilde{\omega}_\nu),$$

where

$$\sigma_\nu(z) = \exp\left\{-\frac{\tilde{\eta}_\nu}{2}z\right\} \frac{\sigma(z + \tilde{\omega}_\nu)}{\sigma(\tilde{\omega}_\nu)}, \quad \zeta_\nu(z) = \zeta(z + \tilde{\omega}_\nu) - \frac{\tilde{\eta}_\nu}{2},$$

and $\tilde{\omega}_\nu, \tilde{\eta}_\nu$ are defined in section 5. We set $G^{(\nu)}$ in the following way:

$$Y(x) = G^{(\nu)} \hat{Y}^{(\nu)}(x)(x - e_\nu) \begin{pmatrix} 0 & \\ & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

And

$$G^{(\nu)} \hat{Y}^{(\nu)}(x) = G_0^{-1} \frac{1}{2} \begin{pmatrix} y_l(z) + y_l(-z) & \frac{y_l(z) - y_l(-z)}{z - \tilde{\omega}_\nu} \\ y_{l+1}(z) + y_{l+1}(-z) & \frac{y_{l+1}(z) - y_{l+1}(-z)}{z - \tilde{\omega}_\nu} \end{pmatrix} \begin{pmatrix} 1 & \\ & \frac{z - \tilde{\omega}_\nu}{\sqrt{x - e_\nu}} \end{pmatrix}$$

$$= G^{(\nu)}(1 + Y_1^{(\nu)}(x - e_\nu) + \dots)$$

$$G^{(\nu)} = G_0^{-1} \begin{pmatrix} y_l(\tilde{\omega}_\nu) & \\ & y_{l+1}(\tilde{\omega}_\nu) \end{pmatrix} \begin{pmatrix} 1 & f_l^{(\nu)} \\ 1 & f_{l+1}^{(\nu)} \end{pmatrix} \begin{pmatrix} 1 & \\ & \sqrt{\frac{2}{\wp''(\tilde{\omega}_\nu)}} \end{pmatrix}.$$

The case $x = \infty, z = 0$ can be calculated in the same way. By direct calculation, we get

$$y_l(0; t, \alpha) = (-1)^{l-1} \frac{\sigma(t + (2l - 1)\alpha)}{\sigma(\alpha)}$$

$$f_l^{(\infty)} = \zeta(t + (2l - 1)\alpha) - (2l - 1)\zeta(\alpha) + t\wp(\alpha).$$

And

$$Y(x) = G^{(\infty)} \hat{Y}^{(\infty)}(x) \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 0 & \\ & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\hat{Y}^{(\infty)}(x) = 1 + O\left(\frac{1}{x}\right)$$

$$G^{(\infty)} = G_0^{-1} \begin{pmatrix} y_l(0) & \\ & y_{l+1}(0) \end{pmatrix} \begin{pmatrix} 1 & f_l^{(\infty)} \\ 1 & f_{l+1}^{(\infty)} \end{pmatrix}.$$

A.3. The τ -function

According to the definition of the τ -function,

$$\frac{\partial}{\partial t} \tau(t) = \wp'(\alpha) \operatorname{tr} Y_1^{(\alpha)} \begin{pmatrix} \frac{1}{2} & \\ & -\frac{1}{2} \end{pmatrix}$$

$$= \zeta(t + 2l\alpha) + \frac{t}{2} \left(\wp(2\alpha) + \frac{\wp''(\alpha)^2}{4\wp'(\alpha)^2} - \frac{\wp'''(\alpha)}{6\wp'(\alpha)} \right) - l \left(\zeta(2\alpha) + \frac{\wp''(\alpha)}{2\wp'(\alpha)} \right).$$

And

$$\tau(t) = \tau_l(t) = \sigma(t + 2l\alpha) \exp h_l(t)$$

$$h_l(t) = \frac{t^2}{4} \left(\wp(2\alpha) + \frac{\wp''(\alpha)^2}{4\wp'(\alpha)^2} - \frac{\wp'''(\alpha)}{6\wp'(\alpha)} \right) - tl \left(\zeta(2\alpha) + \frac{\wp''(\alpha)}{2\wp'(\alpha)} \right).$$

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